Quantum Stabilizers

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1 Introduction

In this report I will give brief overview of quantum stablizers and stabilzers codes. I will generally use Shor's 9 qubit code. Near the end I will show the 5 qubit stablizer and corresponding stabilzer code.

2 Errors

When sending a qubit over a wire there is a possiblity that an error can occur, giving us the incorrect qubit on the other end. We consider an error to be a bit flip, a sign (phase) flip, both a bit and sign flip, or no error occuring. These four errors can be expressed as the Pauli operators below respectivley.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3 Quantum Codes

Because of the no cloning theorem for qubits, we are not able to duplicate a qubit. A quantum code allows us to spread the information contained in a single qubit over multiple entangled qubits. A quantum code is a 2^k dimensional subspace of $\mathbb{C}^{2^n}$ where k is the number of qubits we are encoding and n is the length of the code.[2][3] For example, Shor's 9-qubit code [1] used to encode a single qubit is,

$$\begin{aligned} |0\rangle \to |\bar{0}\rangle &= (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle \to |\bar{1}\rangle &= (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \end{aligned} \tag{3.1}$$

Note that $|\bar{0}\rangle \perp |\bar{1}\rangle$, so $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ spans a 2-dimensional subspace Q of $\mathbb{C}^{2^9}$.

4 Stabilizers

We say that a state $|\psi\rangle$ is stabilized by an operator K if it is a $+1$ eigenstate of K, i.e. $K|\psi\rangle = |\psi\rangle$.

Let $\mathcal{E} = \{E_1 \otimes \ldots \otimes E_n | E_i \in \{X, Y, Z, I\}\}$
Let the quantum code Q be a subspace of $\mathbb{C}^{2^n}$.
Then, the stabilizer of Q is $S = \{M \in \mathcal{E} | Mv = v, \forall v \in Q\}$.

For examples, the stablizer for Shor's 9-qubit code (3.1) is,

M_1	Z	Z	I	I	I	I	I	I	I
M_2	Z	I	Z	I	I	I	I	I	I
M_3	I	I	I	Z	Z	I	I	I	I
M_4	I	I	I	Z	I	Z	I	I	I
M_5	I	I	I	I	I	I	Z	Z	I
M_6	I	I	I	I	I	I	Z	I	Z
M_7	X	X	X	X	X	X	I	I	I
M_8	X	X	X	I	I	I	X	X	X

Stablizers allow us to more easily represent the quantum code, the errors it can detect and correct, and the error correction and detection circuit associated with the quantum code. It also allows us to reduce a multi-qubit space to a single qubit space (assuming we are encoding $k = 1$ qubits).

Note that the stablizer S for Shor's 9-qubit code contains 8 elements. In fact, if our code is of length n and we are encoding k qubits, then S will contain 2^{n-k} elements (4.4), but its minimial representation will contain $n-k$ multiplicative independant generators [3]. Indeed, the stabilzer for Shor's 9-qubit code contains $n-k = 9-1 = 8$ generators. We call these multiplicative independant generators because all other elements in the stablizer can be generated by multiplying some combination of generators. Also, no generator can be generated by multiplying two other generators. For example, consider the following operator,

$$M^* = I \otimes Z \otimes Z \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I$$

M^* stabilizes Shor's 9-qubit code yet it is not included in the minimal stablizer. This is because $M^* = M_1 M_2$.

We can show that $|S| = 2^{n-k}$ by assuming S has $n-k$ generators [3]. Since each $M_i, M_j \in S$ commute (4.5) and $\forall M \in S$ can be generated by multiplying some combination of generators, the size of S will be the sum of all possible combinations of generatoes, i.e

$$|S| = \sum_{i=0}^{n-k} \binom{n-k}{i} \tag{4.1}$$

To find this value we use the Binomial Theorem,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{4.2}$$

Setting $x = y = 1$ in (4.2) we get

$$2^n = \sum_{i=0}^{n} \binom{n}{i} \tag{4.3}$$

We can now use (4.3) to solve (4.1),

$$|S| = \sum_{i=0}^{n-k} \binom{n-k}{i} = 2^{n-k} \tag{4.4}$$

It is interesting to note that any $M_i, M_j \in S$ commute. We can show this using the fact the $Mv = v$ from the definition of a stabilizer.

$$M_i M_j v = M_i v = v = M_j v = M_j M_i v \tag{4.5}$$

We say that if an error E anticommutes with some $M \in S$ then E is detectable. Shor's 9-qubit code can detect a single bit flip, sign flip, or a both a bit and sign flip. Let us suppose a bit flip happens to the first bit in the first 3-qubit block, i.e. $E = X \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I$. Then we can see that,

$$\begin{aligned} EM_1 &= XZ \otimes Z \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \\ &= -ZX \otimes Z \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \\ &= -M_1 E \end{aligned}$$

So, E anticommutes with M_1. It is not difficult to see that if E were to cause a single bit flip in any of the other 3-qubit blocks, then E will anticommute with at least one of $M_1 \ldots M_6$.

Let us assume E causes a sign flip in the second 3-qubit block, i.e $E = I \otimes I \otimes I \otimes Z \otimes I \otimes I \otimes I \otimes I \otimes I$. Then we can see that,

$$\begin{aligned} EM_7 &= X \otimes X \otimes X \otimes ZX \otimes X \otimes X \otimes X \otimes X \otimes Z \\ &= X \otimes X \otimes X \otimes -XZ \otimes X \otimes X \otimes X \otimes X \otimes Z \\ &= -M_7 E \end{aligned}$$

An error E that causes a sign flip in one of the 3-qubit blocks will also anticommute with at least one of M_7 or M_8.

5 Correcting Errors

Let us consider Shor's 9-qubit code (3.1) again. In the the case of a bit flip, the idea behind correcting it would be to compare the first and second qubits in a block and the first and third. Two of the qubits will match, so we just need to change the third to match too. The same idea works for a sign flip, except we are comparing the signs between the 3-qubit blocks.

6 Five Qubit Code

We used Shor's 9-qubit code (3.1) for our examples, but it is not the only quantum error correcting code. There exists a 5-qubit code too. The 5-qubit code is of immense interest because it is the shortest length code that can correct a single error, i.e. both a bit flip and sign flip [1][3].

The stabilizer S for the 5-qubit code is [1]

$$\begin{array}{c|ccccc} M_1 & X & Z & Z & X & I \\ M_2 & I & X & Z & Z & X \\ M_3 & X & I & X & Z & Z \\ M_4 & Z & X & I & X & Z \end{array}$$

To get the first basis codewords we perform the following,

$$\begin{aligned} |\bar{0}\rangle &= \sum_{M \in S} |00000\rangle \\ &= |00000\rangle + M_1 |00000\rangle + M_2 |00000\rangle + M_3 |00000\rangle + M_4 |00000\rangle \\ &+ M_1 M_2 |00000\rangle + M_1 M_3 |00000\rangle + M_1 M_4 |00000\rangle \\ &+ M_2 M_3 |00000\rangle + M_2 M_4 |00000\rangle + M_3 M_4 |00000\rangle \\ &+ M_1 M_2 M_3 |00000\rangle + M_1 M_2 M_4 |00000\rangle + M_1 M_3 M_4 |00000\rangle + M_2 M_3 M_4 |00000\rangle \\ &+ M_1 M_2 M_3 M_4 |00000\rangle \end{aligned}$$

$$|\bar{1}\rangle = X^{\otimes 5} |\bar{0}\rangle$$

The 5-qubit code is a non-CSS (Calderbank-Shor-Steane) code. This means that it mixes Z and X in its generators. By doing this, it is not as straightforward of an approach to create the error correcting circuit [3].

References

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[3] William J. Munro Simon J. Devitt Kae Nemoto. “Quantum Error Correction for Beginners”. In: *Reports on Progress in Physics* 76.7 (2013). DOI: `https://iopscience.iop.org/article/10.1088/0034-4885/76/7/076001`.